The Existence of Solutions to an Even-Order Boundary Value Problem

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The Problem

Consider the fourth order differential equation

$$u^{(4)}(t) = \lambda h\left(t, u(t), u''(t)\right), \tag{1}$$

for $t \in [0,1]$, satisfying the boundary conditions

$$\alpha_1 u(0) - \gamma_1 u(1) = \beta_1 u'(0) - \delta_1 u'(1) = -a, \tag{2}$$
$$\alpha_2 u''(0) - \gamma_2 u''(1) = \beta_2 u'''(0) - \delta_2 u'''(1) = b, \tag{3}$$

where $h : [0,1] \times [0,\infty) \times (-\infty, 0] \to [0,\infty)$ is continuous, $\lambda > 0$, and $a, b \geq 0$; additionally, we require $\alpha_i, \beta_i, \gamma_i, \delta_i > 0$, $\alpha_i > \gamma_i$, $\delta_i > \beta_i$, and $\gamma_i \geq 2\delta_i$ for $i = 1, 2$.

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2. We construct a sequence of lemmas that lead to estimates on a particular operator T.

3. We apply the **Guo-Krasnosel'skii Fixed Point Theorem** three times to show the existence of *at least* three fixed points of T—which, in turn, gives the existence of at least three positive solutions to (1)–(3).

Guo-Krasnosel'skii Fixed Point Theorem

Theorem 1.1: *Let* $(X, \|\cdot\|)$ *be a Banach space, and let* $C \subset X$ *be a cone. Suppose* Ω_1, Ω_2 *are open subsets of* X *satisfying* $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. *If* $T : C \cap (\overline{\Omega_2} \setminus \Omega_1) \to C$ *is a completely continuous operator such that either*

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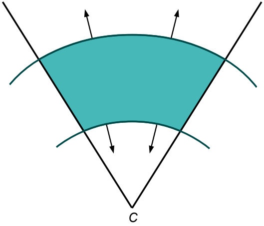
1. $\|Tu\| \leq \|u\|$ *for* $u \in C \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$ *for* $u \in C \cap \partial\Omega_2$,



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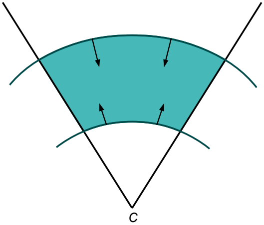

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   or

2. $\|Tu\| \geq \|u\|$ *for* $u \in C \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$ *for* $u \in C \cap \partial\Omega_2$,

then T has a fixed point in $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Substitutions

To convert (1)–(3) into a system of second order differential equations, we make the substitutions

- $u_1 = u$,
- $u_2 = -u''$,
- $g(t, u_1, u_2) = u_2$,
- $f(t, u_1, u_2) = h(t, u_1, -u_2)$.

Substitutions

This gives

$$-u_2''(t) = \lambda f\left(t, u_1, u_2\right), \tag{4}$$

$$-u_1''(t) = g\left(t, u_1, u_2\right), \tag{5}$$

$$\alpha_1 u_1(0) - \gamma_1 u_1(1) = \beta_1 u_1'(0) - \delta_1 u_1'(1) = -a, \tag{6}$$

$$\alpha_2 u_2(0) - \gamma_2 u_2(1) = \beta_2 u_2'(0) - \delta_2 u_2'(1) = -b. \tag{7}$$

Transformations

We now make use of the following transformations:

- $v_1 = u_1 - \dfrac{a}{2\delta_1} t^2 + \dfrac{a\left(2\delta_1 - \gamma_1\right)}{2\delta_1\left(\alpha_1 - \gamma_1\right)}$
- $v_2 = u_2 - \dfrac{a}{2\delta_2} t^2 + \dfrac{a\left(2\delta_2 - \gamma_2\right)}{2\delta_2\left(\alpha_2 - \gamma_2\right)}$

Transformations

Applying the transformations, (4)–(7) we get a system of boundary value problems:

$$-u_2''(t) = \lambda f(t, u_1(t) + Q_1 t^2 + R_1, u_2(t) + Q_2 t^2 + R_2), \quad (8)$$

$$-u_1''(t) = g(t, u_1(t) + Q_1 t^2 + R_1, u_2(t) + Q_2 t^2 + R_2), \quad (9)$$

$$\alpha_i u_i(0) - \gamma_i u_i(1) = \beta_i u_i'(0) - \delta_i u_i'(1) = 0, \quad (10)$$

where $Q_i = \dfrac{a}{2\delta_i}$ and $R_i = -\dfrac{a\,(2\delta_i - \gamma_i)}{2\delta_i\,(\alpha_i - \gamma_i)}$ for $i = 1, 2$.

Transformations

Solutions to (8)–(10) are of the form

$$u_2(t) = \lambda \int_0^1 G_2(t,s) f(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds$$

$$u_1(t) = \int_0^1 G_1(t,s) g(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds,$$

where $G_k(t,s)$ are the Green's functions

$$G_k(t,s) = \frac{1}{M_k N_K} \begin{cases} \delta_k N_k t + \gamma_k M_k s + \gamma_k \beta_k, & 0 \le t \le s \le 1, \\ \beta_k N_k t + \alpha_k M_k s + \gamma_k \beta_k, & 0 \le s \le t \le 1, \end{cases}$$

and $M_k = \delta_k - \beta_k$, $N_k = \alpha_k - \gamma_k$ for $k = 1, 2$.

The Setup

Let $(X, \|\cdot\|)$ be the Banach space
$X = C^1([0,1];\mathbb{R}) \times C^1([0,1];\mathbb{R})$ endowed with the norm

$$\|(u_1, u_2)\| = \|u_1\|_\infty + \|u_2\|_\infty,$$

where $\|u\|_\infty = \sup_{t\in[0,1]} |u(t)|$.

The Setup

Define $C \subset X$ to be the cone

$$C = \{(u_1, u_2) \in X \mid u_i \text{ is nonnegative and concave;} \\ \alpha_i u_i(0) - \gamma_i u_i(1) = \beta_i u_i'(0) - \delta_i u_i'(1) = 0 \text{ for } i = 1, 2\}.$$

Next, let Ω_ρ denote the open set
$\Omega_\rho = \{(u_1, u_2) \in X : \|(u_1, u_2)\| < \rho\}$.

The Setup

Lastly, define $T : X \to X$ to be the operator

$$T(u_1, u_2) = (A_1(u_1, u_2), A_2(u_1, u_2)),$$

where

$$A_2(u_1, u_2)(t) = \lambda \int_0^1 G_2(t, s) f(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds$$

and

$$A_1(u_1, u_2)(t) = \int_0^1 G_1(t, s) g(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds.$$

Solutions of (8)–(10) are fixed points of T.

The Setup

The following lemma gives two properties of T that are needed in order to apply the Guo-Krasnosel'skii Fixed Point Theorem.

Lemma 0. *T is a completely continuous operator and $T : C \to C$.*

Hypotheses

(H0) $f, g : [0,1] \times [0,\infty)^2 \to [0,\infty)$ are continuous functions that are nondecreasing in their last two variables.

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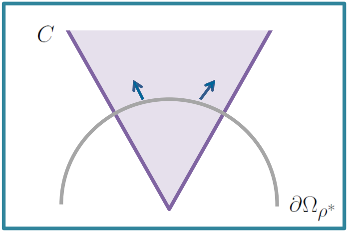
(H1) There exists $\alpha, \beta \in (0,1)$, $\alpha < \beta$, such that, given $(x_1, x_2) \in [0,\infty)^2$ with $x_1 + x_2 \neq 0$, there exists $k > 0$ such that $f(t, x_1, x_2) > k$ for $t \in [\alpha, \beta]$.

Lemma 1

Lemma 1. *Suppose* (H0) *and* (H1) *hold, and let* $\rho^* > 0$. *Then there exists* Λ *such that, for every* $\lambda \geq \Lambda$ *and* $(Q_1, Q_2, R_1, R_2) \in [0, \infty)^4$, *we have*

$$\|T(u_1, u_2)\| \geq \|(u_1, u_2)\|$$

for each $(u_1, u_2) \in C \cap \partial\Omega_{\rho^*}$.

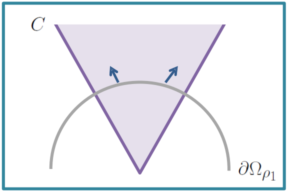


Lemma 2

Lemma 2. *Fix* $\Lambda > 0$, *and suppose* (H0) *and* (H1) *hold. Then, for every* $\lambda \geq \Lambda$ *and* $(Q_1, Q_2, R_1, R_2) \in [0, \infty)^4$, *there exists* $\rho_1 = \rho_1(\Lambda, Q_1, Q_2, R_1, R_2)$ *such that, for every* $\rho \leq \rho_1$, *we have*

$$\|T(u_1, u_2)\| \geq \|(u_1, u_2)\|$$

for each $(u_1, u_2) \in C \cap \partial\Omega_\rho$.



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(H2) Let $z = x_1 + x_2$. Then

$$\lim_{z \to 0^+} \frac{f(t, x_1, x_2)}{z} = 0$$

uniformly for $t \in [0,1]$.

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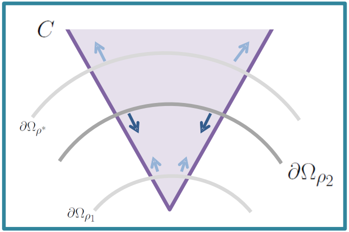
(H3) There exists a $0 < \overline{\zeta} < \frac{2M_1 N_1}{\alpha_1(\delta_1 + \beta_1)}$ and $q > 0$ such that, for all $(x_1, x_2) \in [0,\infty)^2$ with $0 < x_1 + x_2 < q$, we have $g(t, x_1, x_2) \le \overline{\zeta}(x_1 + x_2)$ for each $t \in [0,1]$.

Lemma 3

Lemma 3. *Suppose* (H0), (H2), *and* (H3) *hold, and let* $\rho^* > 0$ *be fixed. Then given* $\lambda > 0$, *there exists* $\rho_2 \in (0, \rho^*)$ *and* $\zeta > 0$ *such that for every* $(Q_1, Q_2, R_1, R_2) \in [0, \infty)^4$ *with* $0 < Q_1 + Q_2 + R_1 + R_2 < \zeta$, *we have*

$$\|T(u_1, u_2)\| \leq \|(u_1, u_2)\|$$

for each $(u_1, u_2) \in C \cap \partial\Omega_{\rho_2}$.



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$$\lim_{z\to\infty} \frac{f(t, x_1, x_2)}{z} = 0$$

uniformly for $t \in [0,1]$.

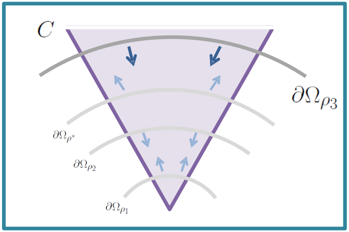
(H5) There exists a $0 < \theta < \frac{2M_1N_1}{\alpha_1(\delta_1+\beta_1)}$ and $r > 0$ such that, for all $(x_1, x_2) \in [0,\infty)^2$ with $x_1 + x_2 > r$, we have $g(t, x_1, x_2) \le \theta(x_1 + x_2)$ for each $t \in [0,1]$.

Lemma 4

Lemma 4. *Suppose* $0 < Q_1 + Q_2 + R_1 + R_2 < \zeta$, *where* $\zeta > 0$ *is given. Suppose further that assumptions* (H0), (H4), *and* (H5) *hold. Then, for every* $\lambda > 0$, *there exists* $\rho_3 = \rho_3(\zeta, \lambda)$ *such that for every* $\rho \geq \rho_3$, *we have*

$$\|T(u_1, u_2)\| \leq \|(u_1, u_2)\|$$

for each $(u_1, u_2) \in C \cap \partial\Omega_\rho$.



Main Result

Theorem 1. *Let f, g satisfy* (H0)–(H5)*. Then there exists $\Lambda > 0$ such that, given any $\lambda \geq \Lambda$, there exists $\zeta > 0$ such that, for every $(Q_1, Q_2, R_1, R_2) \in [0, \infty)^4$ satisfying $0 < Q_1 + Q_2 + R_1 + R_2 < \zeta$, the system (8)–(10) has at least three positive solutions.*