Existence Results for a Class of Even-Order Boundary Value Problems

Daniel Brumley
Advisor: Dr. Britney Hopkins

Department of Mathematics and Statistics
University of Central Oklahoma

April 19, 2018

Problem Statement

This thesis focuses on establishing the existence of positive solutions to even-order boundary value problems (BVPs) of the form

$$u^{(2n)}(t) = \lambda h\left(t, u(t), u''(t), \ldots, u^{(2n-2)}(t)\right), \tag{1}$$

$$\alpha_{i+1} u^{(2i)}(0) - \gamma_{i+1} u^{(2i)}(0) = (-1)^{i+1} a_{i+1}, \qquad i = 0, 1, \ldots, n-1, \tag{2}$$

$$\beta_{i+1} u^{(2i+1)}(1) - \delta_{i+1} u^{(2i+1)}(1) = (-1)^{i+1} a_{i+1}, \qquad i = 0, 1, \ldots, n-1, \tag{3}$$

where $n \geq 2$, $h : [0, 1] \times \prod_{j=0}^{n-1} (-1)^j [0, \infty) \to (-1)^n [0, \infty)$ is continuous, and $\lambda > 0$.

Problem Statement

This thesis focuses on establishing the existence of positive solutions to even-order boundary value problems (BVPs) of the form

$$u^{(2n)}(t) = \lambda h\left(t, u(t), u''(t), \ldots, u^{(2n-2)}(t)\right), \tag{1}$$

$$\alpha_{i+1}u^{(2i)}(0) - \gamma_{i+1}u^{(2i)}(0) = (-1)^{i+1}a_{i+1}, \qquad i = 0, 1, \ldots, n-1, \tag{2}$$

$$\beta_{i+1}u^{(2i+1)}(1) - \delta_{i+1}u^{(2i+1)}(1) = (-1)^{i+1}a_{i+1}, \qquad i = 0, 1, \ldots, n-1, \tag{3}$$

where $n \geq 2$, $h : [0,1] \times \prod_{j=0}^{n-1}(-1)^j[0,\infty) \to (-1)^n[0,\infty)$ is continuous, and $\lambda > 0$.

In particular, for $i = 0, 1, \ldots, n-1$, we require $\alpha_{i+1}, \beta_{i+1}, \gamma_{i+1}, \delta_{i+1} > 0$ and consider in tandem the cases

(i) $a_{i+1} > 0$ and (ii) $a_{i+1} < 0$.

Problem Statement

This thesis focuses on establishing the existence of positive solutions to even-order boundary value problems (BVPs) of the form

$$u^{(2n)}(t) = \lambda h\left(t, u(t), u''(t), \ldots, u^{(2n-2)}(t)\right), \tag{1}$$

$$\alpha_{i+1} u^{(2i)}(0) - \gamma_{i+1} u^{(2i)}(0) = (-1)^{i+1} a_{i+1}, \qquad i = 0, 1, \ldots, n-1, \tag{2}$$

$$\beta_{i+1} u^{(2i+1)}(1) - \delta_{i+1} u^{(2i+1)}(1) = (-1)^{i+1} a_{i+1}, \qquad i = 0, 1, \ldots, n-1, \tag{3}$$

where $n \geq 2$, $h : [0,1] \times \prod_{j=0}^{n-1} (-1)^j [0, \infty) \to (-1)^n [0, \infty)$ is continuous, and $\lambda > 0$.

In particular, for $i = 0, 1, \ldots, n-1$, we require $\alpha_{i+1}, \beta_{i+1}, \gamma_{i+1}, \delta_{i+1} > 0$ and consider in tandem the cases

(i) $a_{i+1} > 0$ and (ii) $a_{i+1} < 0$.

For the sake of brevity, we focus exclusively on case (i) during this talk.

Motivation

Why is this work important?

Motivation: Practical

- From a practical standpoint, the study of multiple solutions to BVPs is important to the modeling of various physical phenomena.

Motivation: Practical

- From a practical standpoint, the study of multiple solutions to BVPs is important to the modeling of various physical phenomena.
  - For instance, Cohen studied the multiplicity of solutions to the BVP

$$\begin{aligned} &\beta u'' - u' + f(u) = 0, \qquad 0 \le t \le 1, \\ &u'(0) - \alpha u(0) = 0, u'(1) = 0, \end{aligned}$$

which occurs in the modeling of a certain chemical reactor.

Motivation: Practical

- From a practical standpoint, the study of multiple solutions to BVPs is important to the modeling of various physical phenomena.
  - For instance, Cohen studied the multiplicity of solutions to the BVP

$$\begin{aligned} &\beta u'' - u' + f(u) = 0, \qquad 0 \leq t \leq 1, \\ &u'(0) - \alpha u(0) = 0, u'(1) = 0, \end{aligned}$$

    which occurs in the modeling of a certain chemical reactor.
  - Argawal addressed uniqueness issues to BVPs of the form

$$\begin{aligned} &u^{(4)} = f(t, u, u', u'', u'''), \qquad a \leq t \leq b \\ &u(a) = A, u'(a) = B, u(b) = C, u''(b) = D, \end{aligned}$$

    motivated by problems arising in beam analysis.

Motivation: Theoretical

- A particularly fruitful approach to proving the existence of multiple solutions hinges on transforming a higher order problem into a system of second-order differential equations of the form $u''(t) = f(t, u(t))$ satisfying homogeneous boundary conditions and observing that solutions to this problem are just fixed points of the operator

$$Tu = \int_0^1 G(t,s) f(s, u(s))\, ds,$$

where G is the Green's function corresponding to the specified homogeneous boundary conditions.

Motivation: Theoretical

- A particularly fruitful approach to proving the existence of multiple solutions hinges on transforming a higher order problem into a system of second-order differential equations of the form $u''(t) = f(t, u(t))$ satisfying homogeneous boundary conditions and observing that solutions to this problem are just fixed points of the operator

$$Tu = \int_0^1 G(t,s)f(s,u(s))ds,$$

where G is the Green's function corresponding to the specified homogeneous boundary conditions.
- As a result, various fixed point theorems have been utilized or proposed to address existence/uniqueness issues, which makes the study of multiple solutions to BVPs an area of significant theoretical importance.

Motivation: Historical

- One of the more important fixed point theorems to arise in the past sixty years in the study of solutions to BVPs is attributable to Krasnosel'skii.

Motivation: Historical

- One of the more important fixed point theorems to arise in the past sixty years in the study of solutions to BVPs is attributable to Krasnosel'skii.

- His work established a fixed point result for operators acting on **cones**, which are nonempty, closed, convex subsets C of a Banach space X such that

Motivation: Historical

- One of the more important fixed point theorems to arise in the past sixty years in the study of solutions to BVPs is attributable to Krasnosel'skii.

- His work established a fixed point result for operators acting on **cones**, which are nonempty, closed, convex subsets C of a Banach space X such that
   (i) if $x \in C$, then $\lambda x \in C$ for all real $\lambda > 0$;

Motivation: Historical

- One of the more important fixed point theorems to arise in the past sixty years in the study of solutions to BVPs is attributable to Krasnosel'skii.

- His work established a fixed point result for operators acting on **cones**, which are nonempty, closed, convex subsets C of a Banach space X such that
  (i) if $x \in C$, then $\lambda x \in C$ for all real $\lambda > 0$;
  (ii) if $x \in C$ and $-x \in C$, then $x = 0$.

Motivation: Historical

- One of the more important fixed point theorems to arise in the past sixty years in the study of solutions to BVPs is attributable to Krasnosel'skii.

- His work established a fixed point result for operators acting on **cones**, which are nonempty, closed, convex subsets C of a Banach space X such that
  (i) if $x \in C$, then $\lambda x \in C$ for all real $\lambda > 0$;
  (ii) if $x \in C$ and $-x \in C$, then $x = 0$.

- An extension was later formulated by Guo. This more general result is known as the **Guo-Krasnosel'skii Fixed Point Theorem**.

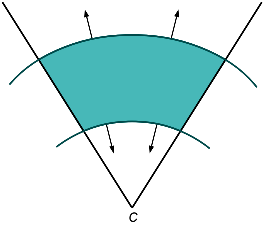
Guo-Krasnosel'skii Fixed Point Theorem

Theorem. *Let* $(X, \|\cdot\|)$ *be a Banach space, and let* $C \subset X$ *be a cone. Suppose* Ω_1, Ω_2 *are open subsets of* X *satisfying* $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$*. If* $T : C \cap (\overline{\Omega}_2 - \Omega_1) \to C$ *is a completely continuous operator such that either*

Guo-Krasnosel'skii Fixed Point Theorem

Theorem. *Let $(X, \|\cdot\|)$ be a Banach space, and let $C \subset X$ be a cone. Suppose Ω_1, Ω_2 are open subsets of X satisfying $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. If $T : C \cap (\overline{\Omega}_2 - \Omega_1) \to C$ is a completely continuous operator such that either*

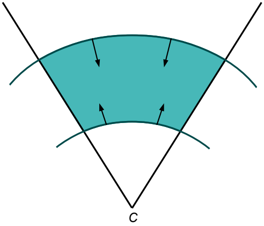
1. $\|Tu\| \leq \|u\|$ *for* $u \in C \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$ *for* $u \in C \cap \partial\Omega_2$,



Guo-Krasnosel'skii Fixed Point Theorem

Theorem. *Let $(X, \|\cdot\|)$ be a Banach space, and let $C \subset X$ be a cone. Suppose Ω_1, Ω_2 are open subsets of X satisfying $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. If $T : C \cap (\overline{\Omega}_2 - \Omega_1) \to C$ is a completely continuous operator such that either*

2. $\|Tu\| \geq \|u\|$ *for* $u \in C \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$ *for* $u \in C \cap \partial\Omega_2$,



Guo-Krasnosel'skii Fixed Point Theorem

Theorem. *Let $(X, \|\cdot\|)$ be a Banach space, and let $C \subset X$ be a cone. Suppose Ω_1, Ω_2 are open subsets of X satisfying $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. If $T : C \cap (\overline{\Omega}_2 - \Omega_1) \to C$ is a completely continuous operator such that either*

1. $\|Tu\| \leq \|u\|$ *for* $u \in C \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$ *for* $u \in C \cap \partial\Omega_2$,

   or

2. $\|Tu\| \geq \|u\|$ *for* $u \in C \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$ *for* $u \in C \cap \partial\Omega_2$,

then T has a fixed point in $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Motivation: Historical (cont'd)

- By utilizing the Guo-Krasnosel'skii Fixed Point Theorem, Marcos, Lorca, and Ubilla demonstrated the existence of at least three positive solutions to the BVP

$$u^{(4)} = \lambda h(t, u, u''), \qquad t \in (0, 1),$$
$$u(0) = u''(0) = 0, u(1) = a, u''(1) = -b.$$

Motivation: Historical (cont'd)

- By utilizing the Guo-Krasnosel'skii Fixed Point Theorem, Marcos, Lorca, and Ubilla demonstrated the existence of at least three positive solutions to the BVP

$$u^{(4)} = \lambda h(t, u, u''), \qquad t \in (0, 1),$$
$$u(0) = u''(0) = 0, u(1) = a, u''(1) = -b.$$

- Hopkins later expanded upon this work in her doctoral dissertation and subsequent papers by generalizing the BVP above to arbitrary order and considering analogous problems on both continuous and discrete domains.

Motivation: Historical (cont'd)

- By utilizing the Guo-Krasnosel'skii Fixed Point Theorem, Marcos, Lorca, and Ubilla demonstrated the existence of at least three positive solutions to the BVP

$$u^{(4)} = \lambda h(t, u, u''), \qquad t \in (0,1),$$
$$u(0) = u''(0) = 0, u(1) = a, u''(1) = -b.$$

- Hopkins later expanded upon this work in her doctoral dissertation and subsequent papers by generalizing the BVP above to arbitrary order and considering analogous problems on both continuous and discrete domains.
- This thesis is an outgrowth of continued investigations (with Drs. Fulkerson, Hopkins, Karber, and Milligan) into the multiplicity of solutions to various classes of even-order boundary value problems couched within this established framework.

The Method

The method common to all this work can be outlined as follows:

The Method

The method common to all this work can be outlined as follows:

1. Transform the boundary value problem into a system of second-order differential equations satisfying homogeneous boundary conditions.

The Method

The method common to all this work can be outlined as follows:

1. Transform the boundary value problem into a system of second-order differential equations satisfying homogeneous boundary conditions.
2. Define a cone C and a completely continuous, cone invariant operator T in such a way that fixed points of T (over C) correspond to solutions to the transformed problem.

The Method

The method common to all this work can be outlined as follows:

1. Transform the boundary value problem into a system of second-order differential equations satisfying homogeneous boundary conditions.
2. Define a cone C and a completely continuous, cone invariant operator T in such a way that fixed points of T (over C) correspond to solutions to the transformed problem.
3. Construct a sequence of lemmas that lead to contraction and expansion estimates for T over nested open subsets of C.

The Method

The method common to all this work can be outlined as follows:

1. Transform the boundary value problem into a system of second-order differential equations satisfying homogeneous boundary conditions.
2. Define a cone C and a completely continuous, cone invariant operator T in such a way that fixed points of T (over C) correspond to solutions to the transformed problem.
3. Construct a sequence of lemmas that lead to contraction and expansion estimates for T over nested open subsets of C.
4. Apply the Guo-Krasnosel'skii Fixed Point Theorem three times to show the existence of **at least** three fixed points of T and, hence, at least three positive solutions to the transformed problem. The existence of multiple positive solutions to the original problem can then be established as a corollary.

The Method

We now proceed to apply this method to the system (1)–(3) to obtain at least three positive solutions.

Substitutions and Transformations

1. Transform the boundary value problem (1)–(3) into a system of second-order differential equations satisfying homogeneous boundary conditions.

Substitutions

For $t \in [0,1]$, we apply the substitutions

$$u_{i+1}(t) = (-1)^i u^{(2i)}(t), \qquad i = 0, 1, \ldots, n-1,$$
$$u_{i+1}(t) = g_i(t, u_1, u_2, \ldots, u_n), \qquad i = 1, 2, \ldots, n-1,$$
$$f\,(t, u_1, u_2, \ldots, u_n) = h\left(t, u_1, -u_2, \ldots, (-1)^{n+1}\, u_n\right).$$

Substitutions

This gives

$$
\begin{aligned}
-u_n''(t) &= \lambda f\left(t, u_1, u_2, \ldots, u_n\right), && && (4)\\
-u_i''(t) &= g_i\left(t, u_1, u_2, \ldots, u_n\right), && i = 1, 2, \ldots, n-1, && (5)\\
\alpha_i u_i(0) - \gamma_i u_i(1) &= \beta_i u_i'(0) - \delta_i u_i'(1) = -a_i, && i = 1, 2, \ldots, n. && (6)
\end{aligned}
$$

Substitutions

This gives

$$-u_n''(t) = \lambda f\left(t, u_1, u_2, \ldots, u_n\right), \tag{4}$$

$$-u_i''(t) = g_i\left(t, u_1, u_2, \ldots, u_n\right), \qquad i = 1, 2, \ldots, n-1, \tag{5}$$

$$\alpha_i u_i(0) - \gamma_i u_i(1) = \beta_i u_i'(0) - \delta_i u_i'(1) = -a_i, \qquad i = 1, 2, \ldots, n. \tag{6}$$

The choice of substitutions combined with the sign changing properties of h imply that f and g are nonnegative.

Substitutions

This gives

$$
\begin{aligned}
-u_n''(t) &= \lambda f\left(t, u_1, u_2, \ldots, u_n\right), && && (4)\\
-u_i''(t) &= g_i\left(t, u_1, u_2, \ldots, u_n\right), && i = 1, 2, \ldots, n-1, && (5)\\
\alpha_i u_i(0) - \gamma_i u_i(1) &= \beta_i u_i'(0) - \delta_i u_i'(1) = -a_i, && i = 1, 2, \ldots, n. && (6)
\end{aligned}
$$

The choice of substitutions combined with the sign changing properties of h imply that f and g are nonnegative.

Consequently, $u_1, u_2, \ldots, u_n$ are nonnegative and concave.

Transformations

To transform (4)–(6) into a system with homogeneous boundary conditions, we make use of the ansatz

$$\overline{u}_i'(t) = u_i'(t) - \frac{a_i}{\delta_i} t. \qquad (*)$$

Transformations

To transform (4)–(6) into a system with homogeneous boundary conditions, we make use of the ansatz

$$\overline{u}_i'(t) = u_i'(t) - \frac{a_i}{\delta_i} t. \qquad (*)$$

Note that (*) satisfies the boundary conditions

$$\beta_i \overline{u}_i'(0) - \delta_i \overline{u}_i'(1) = 0$$

for $i = 1, 2, \ldots, n$.

Transformations

Integrating both sides of (*) with respect to t gives

$$\overline{u}_i(t) = u_i(t) - \frac{a_i}{2\delta_i}t^2 + C_i, \qquad (**)$$

where $C_i \in \mathbb{R}$ for $i = 1, 2, \ldots, n$.

Transformations

Integrating both sides of (*) with respect to t gives

$$\overline{u}_i(t) = u_i(t) - \frac{a_i}{2\delta_i} t^2 + C_i, \qquad (**)$$

where $C_i \in \mathbb{R}$ for $i = 1, 2, \ldots, n$.

We would like to choose C_i in (**) so that the remaining boundary conditions

$$\alpha_i \overline{u}_i(0) - \gamma_i \overline{u}_i(1) = 0$$

are satisfied for $i = 1, 2, \ldots, n$.

Transformations

That is, we need

$$0 = \alpha_i \overline{u}_i(0) - \gamma_i \overline{u}_i(1) = \alpha_i \left[u_i(0) + C_i\right] - \gamma_i \left[u_i(1) - \frac{a_i}{2\delta_i} + C_i\right]$$

Transformations

That is, we need

$$0 = \alpha_i \overline{u}_i(0) - \gamma_i \overline{u}_i(1) = \alpha_i \left[u_i(0) + C_i\right] - \gamma_i \left[u_i(1) - \frac{a_i}{2\delta_i} + C_i\right]$$
$$= \left[\alpha_i u_i\left(0\right) - \gamma_i u_i\left(1\right)\right] + \frac{a_i \gamma_i}{2\delta_i} + \left(\alpha_i - \gamma_i\right) C_i$$

Transformations

That is, we need

$$
\begin{aligned}
0 = \alpha_i \overline{u}_i(0) - \gamma_i \overline{u}_i(1) &= \alpha_i \left[u_i(0) + C_i\right] - \gamma_i \left[u_i(1) - \frac{a_i}{2\delta_i} + C_i\right] \\
&= \left[\alpha_i u_i(0) - \gamma_i u_i(1)\right] + \frac{a_i \gamma_i}{2\delta_i} + (\alpha_i - \gamma_i)\, C_i \\
&= -a_i + \frac{a_i \gamma_i}{2\delta_i} + (\alpha_i - \gamma_i)\, C_i,
\end{aligned}
$$

Transformations

That is, we need

$$\begin{aligned} 0 = \alpha_i \overline{u}_i(0) - \gamma_i \overline{u}_i(1) &= \alpha_i \left[u_i(0) + C_i\right] - \gamma_i \left[u_i(1) - \frac{a_i}{2\delta_i} + C_i\right] \\ &= \left[\alpha_i u_i(0) - \gamma_i u_i(1)\right] + \frac{a_i \gamma_i}{2\delta_i} + (\alpha_i - \gamma_i) C_i \\ &= -a_i + \frac{a_i \gamma_i}{2\delta_i} + (\alpha_i - \gamma_i) C_i, \end{aligned}$$

and so we must have $C_i = \frac{a_i(2\delta_i - \gamma_i)}{2\delta_i(\alpha_i - \gamma_i)}$.

Transformations

Thus, by setting

$$\overline{u}_i(t) = u_i(t) - \frac{a_i}{2\delta_i}t^2 + \frac{a_i\,(2\delta_i - \gamma_i)}{2\delta_i\,(\alpha_i - \gamma_i)},$$

we obtain functions that simultaneously satisfy the homogeneous boundary conditions

$$\alpha_i\overline{u}_i(0) - \gamma_i\overline{u}_i(1) = \beta_i\overline{u}_i'(0) - \delta_i\overline{u}_i'(1) = 0.$$

Transformations

Applying the previous transformations to (4)–(6), we get the system of boundary value problems

$$-u_n'' = \lambda f(t, u_1 + Q_1 t^2 + R_1, \ldots, u_n + Q_n t^2 + R_n), \tag{7}$$

$$-u_i'' = g_i(t, u_1 + Q_1 t^2 + R_1, \ldots, u_n + Q_n t^2 + R_n), \quad i = 1, 2, \ldots, n-1, \tag{8}$$

$$\alpha_i u_i(0) - \gamma_i u_i(1) = \beta_i u_i'(0) - \delta_i u_i'(1) = 0, \quad i = 1, 2, \ldots, n, \tag{9}$$

where $Q_i = \frac{a_i}{2\delta_i}$ and $R_i = -\frac{a_i(2\delta_i - \gamma_i)}{2\delta_i(\alpha_i - \gamma_i)}$ for $i = 1, 2, \ldots, n$.

Transformations

Solutions to (7)–(9) are of the form

$$u_n(t) = \lambda \int_0^1 G_n(t,s) f(s, u_1(s) + Q_1 s^2 + R_1, \dots, u_n(s) + Q_n s^2 + R_n) ds$$

$$u_i(t) = \int_0^1 G_i(t,s) g_i(s, u_1(s) + Q_1 s^2 + R_1, \dots, u_n(s) + Q_n s^2 + R_n) ds,$$

for $i = 1, 2, \dots, n-1$ and where $G_k(t,s)$ is the Green's function

$$G_k(t,s) = \frac{1}{M_k N_k} \begin{cases} \delta_k N_k t + \gamma_k M_k s + \gamma_k \beta_k, & 0 \le t \le s \le 1, \\ \beta_k N_k t + \alpha_k M_k s + \gamma_k \beta_k, & 0 \le s \le t \le 1, \end{cases}$$

with $M_k = \delta_k - \beta_k$, $N_k = \alpha_k - \gamma_k$ for $k = 1, 2, \dots, n$.

Transformations

To ensure **positive** solutions to (1)–(3)—or, equivalently, (7)–(9)—we require

$$\alpha_i > \gamma_i \text{ and } \delta_i > \beta_i \text{ for } i = 1, 2, \ldots, n$$

so that

$$(M_i N_i)^{-1} (\delta_i N_i t + \gamma_i M_i s + \gamma_i \beta_i) > 0$$

and

$$(M_i N_i)^{-1} (\beta_i N_i t + \alpha_i M_i s + \gamma_i \beta_i) > 0,$$

from which it follows the $G_i(t, s)$ and, hence, the solutions to (7)–(9) are positive.

Cone and Operator

2. Define a cone C and a completely continuous, cone preserving operator T in such a way that fixed points of T (over C) correspond to solutions to the transformed problem.

Cone and Operator

Let $(X, \|\cdot\|)$ be the Banach space $X = \prod_{i=1}^{n} C^1([0,1];\mathbb{R})$ endowed with the norm

$$\|(u_1, \ldots, u_n)\| = \sum_{i=1}^{n} \|u_i\|_\infty,$$

where $\|u\|_\infty = \sup_{t\in[0,1]} |u(t)|$.

Cone and Operator

Define $C \subset X$ to be the cone

$$C = \{(u_1, \ldots, u_n) \in X \mid u_i \text{ is nonnegative and concave;} \\ \alpha_i u_i(0) - \gamma_i u_i(1) = \beta_i u_i'(0) - \delta_i u_i'(1) = 0 \text{ for } i = 1, 2, \ldots, n.\}.$$

Cone and Operator

Define $C \subset X$ to be the cone

$$C = \{(u_1, \ldots, u_n) \in X \mid u_i \text{ is nonnegative and concave;}$$
$$\alpha_i u_i(0) - \gamma_i u_i(1) = \beta_i u_i'(0) - \delta_i u_i'(1) = 0 \text{ for } i = 1, 2, \ldots, n.\}.$$

The verification that C is a cone is straightforward and left as an exercise. :)

Cone and Operator

Define $C \subset X$ to be the cone

$$C = \{(u_1, \ldots, u_n) \in X \mid u_i \text{ is nonnegative and concave;} \\ \alpha_i u_i(0) - \gamma_i u_i(1) = \beta_i u_i'(0) - \delta_i u_i'(1) = 0 \text{ for } i = 1, 2, \ldots, n.\}.$$

The verification that C is a cone is straightforward and left as an exercise. :)

We also let Ω_ρ denote the open set

$$\Omega_\rho = \{(u_1, \ldots, u_n) \in X : \|(u_1, \ldots, u_n)\| < \rho\},$$

and write $\partial\Omega_\rho$ for the boundary of Ω_ρ, that is,

$$\partial\Omega_\rho = \{(u_1, \ldots, u_n) \in X : \|(u_1, \ldots, u_n)\| = \rho\}.$$

Cone and Operator

Finally, define $T : X \to X$ to be the operator

$$T(u_1, \dots, u_n) = (A_1(u_1, \dots, u_n), \dots, A_n(u_1, \dots, u_n)),$$

where

$$A_n(u_1, \dots, u_n)(t) =$$
$$\lambda \int_0^1 G_n(t,s) f(s, u_1(s) + Q_1 s^2 + R_1, \dots, u_n(s) + Q_n s^2 + R_n) ds,$$
$$A_i(u_1, \dots, u_n)(t) =$$
$$\int_0^1 G_i(t,s) g_i(s, u_1(s) + Q_1 s^2 + R_1, \dots, u_n(s) + Q_n s^2 + R_n) ds, \quad i = 1, 2, \dots, n-1,$$

with λ and G_i defined as above and

$$(Q_1, \dots, Q_n, R_1, \dots, R_n) \in [0, \infty)^{2n}.$$

Cone and Operator

By design, the fixed points of T (over C), if any, are solutions to a system that is similar to (7)–(9) in form but in which the only constraints on $Q_1, \ldots, Q_n, R_1, \ldots, R_n$ are nonnegativity.

Cone and Operator

By design, the fixed points of T (over C), if any, are solutions to a system that is similar to (7)–(9) in form but in which the only constraints on $Q_1, \ldots, Q_n, R_1, \ldots, R_n$ are nonnegativity.

This has the obvious advantage of generality; the disadvantage is that it might make this talk slightly more confusing as a result.

Cone and Operator

By design, the fixed points of T (over C), if any, are solutions to a system that is similar to (7)–(9) in form but in which the only constraints on $Q_1, \ldots, Q_n, R_1, \ldots, R_n$ are nonnegativity.

This has the obvious advantage of generality; the disadvantage is that it might make this talk slightly more confusing as a result.

For the sake of clarity, we refer to this more general system as (7*)–(9*).

Cone and Operator

The following hypothesis will be the backbone for all our later work:

(H0) For $i = 1, 2, \ldots, n-1$, the functions $f, g_i : [0,1] \times [0,\infty)^n \to [0,\infty)$ are continuous and nondecreasing in their last n variables.

Cone and Operator

The following hypothesis will be the backbone for all our later work:

(H0) For $i = 1, 2, \ldots, n-1$, the functions $f, g_i : [0,1] \times [0,\infty)^n \to [0,\infty)$ are continuous and nondecreasing in their last n variables.

The addition of (H0) introduces constraints on the functions $f, g_1, \ldots, g_n$ and the constants $Q_1, \ldots, Q_n, R_1, \ldots, R_n$ in (7*)–(9*) that may or may not hold for their counterparts in (7)–(9).

Cone and Operator

The following hypothesis will be the backbone for all our later work:

(H0) For $i = 1, 2, \ldots, n-1$, the functions $f, g_i : [0,1] \times [0,\infty)^n \to [0,\infty)$ are continuous and nondecreasing in their last n variables.

The addition of (H0) introduces constraints on the functions $f, g_1, \ldots, g_n$ and the constants $Q_1, \ldots, Q_n, R_1, \ldots, R_n$ in (7*)–(9*) that may or may not hold for their counterparts in (7)–(9).

As a result, it is possible the natural correspondence between the two systems will be compromised, unless we appropriately constrain the function h and parameters $\alpha_i, \beta_i, \gamma_i, \delta_i$ of (1)–(3).

Additional Constraints

Incidentally, the continuity and nonnegativity properties of the functions f, g_i follow directly from the continuity and "sign changing" properties of h coupled with the choice of substitutions/transformations made earlier.

Additional Constraints

Incidentally, the continuity and nonnegativity properties of the functions f, g_i follow directly from the continuity and "sign changing" properties of h coupled with the choice of substitutions/transformations made earlier.

The nondecreasing properties cannot be similarly deduced, so we make the following assumption on h:

h is nondecreasing in its $(2j)$th variables,

and nonincreasing in its $(2j+1)$th variables for $j = 1, 2, \ldots, n$.

Additional Constraints

The constraints on the parameters are more subtle:

Additional Constraints

The constraints on the parameters are more subtle:

- All our subsequent work will take place in the cone C, where the functions $u_1, u_2, \ldots, u_n$ are assumed to be nonnegative.

Additional Constraints

The constraints on the parameters are more subtle:

- All our subsequent work will take place in the cone C, where the functions $u_1, u_2, \ldots, u_n$ are assumed to be nonnegative.
- Moreover, we have $(Q_1, \ldots, Q_n, R_1, \ldots, R_n) \in [0, \infty)^{2n}$ and $f, g_i : [0, 1] \times [0, \infty)^n \to [0, \infty)$ for $i = 1, 2, \ldots, n-1$.

Additional Constraints

We must therefore have

$$0 \le \min_{\substack{u_i \in C, \\ s \in [0,1]}} \left\{ u_i(s) + Q_i s^2 + R_i \right\} = R_i$$

for $i = 1, 2, \ldots, n$.

Additional Constraints

We must therefore have

$$0 \leq \min_{\substack{u_i \in C, \\ s \in [0,1]}} \left\{ u_i(s) + Q_i s^2 + R_i \right\} = R_i$$

for $i = 1, 2, \ldots, n$.

In the transformed system (7)–(9), this amounts to

$$0 \leq R_i = \frac{-a_i \left(2\delta_i - \gamma_i\right)}{2\delta_i \left(\alpha_i - \gamma_i\right)},$$

Additional Constraints

We must therefore have

$$0 \leq \min_{\substack{u_i \in C, \\ s \in [0,1]}} \left\{ u_i(s) + Q_i s^2 + R_i \right\} = R_i$$

for $i = 1, 2, \ldots, n$.

In the transformed system (7)–(9), this amounts to

$$0 \leq R_i = \frac{-a_i \left(2\delta_i - \gamma_i\right)}{2\delta_i \left(\alpha_i - \gamma_i\right)},$$

from which get the requirement in (1)–(3) that

$$\boxed{2\delta_i \leq \gamma_i \text{ for } i = 1, 2, \ldots, n.}$$

Lemma A

The following preliminary lemma establishes the completely continuous and cone preserving properties of T.

Lemma A. *Suppose (H0) holds. Then T is a completely continuous operator such that $T(C) \subseteq C$.*

Some Bounds

The following bounds will be needed not only in the proof of Lemma A but also the proofs of subsequent lemmas:

$$\max_{t\in[0,1]} \int_0^1 G_i(t,s)\,ds = \frac{\alpha_i\left(\delta_i+\beta_i\right)}{2M_iN_i}, \qquad i=1,2,\ldots,n, \tag{10}$$

and

$$\max_{t\in[0,1]} \int_0^1 \frac{\partial}{\partial t} G_i(t,s)\,ds = \frac{\delta_i}{M_i}, \qquad i=1,2,\ldots,n. \tag{11}$$

Lemma A: Proof Outline

- That T is cone preserving follows immediately from the definitions.

Lemma A: Proof Outline

- That T is cone preserving follows immediately from the definitions.
- The completely continuous property of T can be established by a standard argument utilizing the Arzela-Ascoli Theorem.

Lemmas

3. Construct a sequence of lemmas that lead to contraction and expansion estimates for T over nested open subsets of C.

Lemmas 1 and 2: Hypotheses

The first two lemmas lead to expansion estimates on T and require the following hypotheses:

Lemmas 1 and 2: Hypotheses

The first two lemmas lead to expansion estimates on T and require the following hypotheses:

(H0) For $i = 1, 2, \ldots, n-1$, the functions $f, g_i : [0,1] \times [0,\infty)^n \to [0,\infty)$ are continuous and nondecreasing in their last n variables.

Lemmas 1 and 2: Hypotheses

The first two lemmas lead to expansion estimates on T and require the following hypotheses:

(H0) For $i = 1, 2, \ldots, n-1$, the functions $f, g_i : [0,1] \times [0,\infty)^n \to [0,\infty)$ are continuous and nondecreasing in their last n variables.

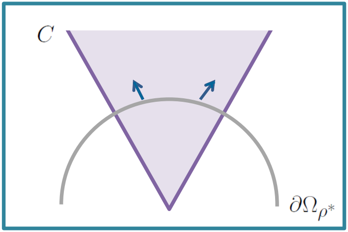
(H1) There exists $\alpha, \beta \in (0,1)$, $\alpha < \beta$, such that, given $(x_1, \ldots, x_n) \in [0,\infty)^n$ with $\sum_{i=1}^n x_i \neq 0$, there exists $\kappa > 0$ such that $f(t, x_1, \ldots, x_n) > \kappa$ for $t \in [\alpha, \beta]$.

Lemma 1

Lemma 1. *Suppose* (H0) *and* (H1) *hold, and let* $\rho^* > 0$. *Then there exists* Λ *such that, for every* $\lambda \geq \Lambda$ *and* $(Q_1, \ldots, Q_n, R_1, \ldots, R_n) \in [0, \infty)^{2n}$, *we have*

$$\|T(u_1, \ldots, u_n)\| \geq \|(u_1, \ldots, u_n)\|$$

for each $(u_1, \ldots, u_n) \in C \cap \partial\Omega_{\rho^*}$.



Lemma B

The proof of Lemma 1 depends on the following lemma:

Lemma B. *Let $u(t)$ be a nonnegative, concave function that is continuous on $[0,1]$. Then, for all $\alpha, \beta \in (0,1)$ with $\alpha < \beta$, we have*

$$\inf_{t\in[\alpha,\beta]} u(t) \geq \alpha\,(1-\beta)\,\|u\|_\infty .$$

Lemma 1: Proof

Let $\rho^* > 0$ and $(u_1, \ldots, u_n) \in C \cap \partial\Omega_{\rho^*}$.

Lemma 1: Proof

Let $\rho^* > 0$ and $(u_1, \ldots, u_n) \in C \cap \partial\Omega_{\rho^*}$.

Assume α and β are as in (H1), and set $r = \alpha(1 - \beta)$.

Lemma 1: Proof

Let $\rho^* > 0$ and $(u_1, \ldots, u_n) \in C \cap \partial\Omega_{\rho^*}$.

Assume α and β are as in (H1), and set $r = \alpha(1-\beta)$.

Define

$$K = \inf\left\{ \frac{f(t, rc_1, \ldots, rc_n)}{r\sum_{i=1}^n c_i} : c_1, \ldots, c_n \geq 0, \sum_{i=1}^n c_i = p^*, t \in [\alpha, \beta] \right\}.$$

Lemma 1: Proof

Let $\rho^* > 0$ and $(u_1, \ldots, u_n) \in C \cap \partial\Omega_{\rho^*}$.

Assume α and β are as in (H1), and set $r = \alpha(1-\beta)$.

Define

$$K = \inf\left\{ \frac{f(t, rc_1, \ldots, rc_n)}{r\sum_{i=1}^{n} c_i} : c_1, \ldots, c_n \geq 0, \sum_{i=1}^{n} c_i = p^*, t \in [\alpha, \beta] \right\}.$$

The existence of a positive K follows from assumption (H1).

Lemma 1: Proof

Now set $\Lambda \geq \left[Kr \int_{\alpha}^{\beta} G_n(1,s)ds \right]^{-1}$.

Lemma 1: Proof

Now set $\Lambda \geq \left[Kr \int_{\alpha}^{\beta} G_n(1,s)ds\right]^{-1}$.

Utilizing Lemma B, we know that

$$u_i(t) + Q_i t^2 + R_i \geq \inf_{t\in[\alpha,\beta]} u_i(t) \geq r\|u_i\|_\infty$$

for $t \in [\alpha, \beta]$ and $i = 1, 2, \ldots, n-1$.

Lemma 1: Proof

Pairing the above with the nondecreasing properties of f, it follows that

Lemma 1: Proof

Pairing the above with the nondecreasing properties of f, it follows that

$$\|T(u_1, \ldots, u_n)\| \geq \|A_n(u_1, \ldots, u_n)\|_\infty$$

Lemma 1: Proof

Pairing the above with the nondecreasing properties of f, it follows that

$$\begin{aligned}\|T(u_1,\ldots,u_n)\| &\geq \|A_n(u_1,\ldots,u_n)\|_\infty \\ &\geq \lambda \int_0^1 G_n(1,s) f(s, u_1(s) + Q_1 s^2 + R_1, \ldots, u_n(s) + Q_n s^2 + R_n) ds\end{aligned}$$

Lemma 1: Proof

Pairing the above with the nondecreasing properties of f, it follows that

$$
\begin{aligned}
\|T(u_1,\dots,u_n)\| &\geq \|A_n(u_1,\dots,u_n)\|_\infty \\
&\geq \lambda \int_0^1 G_n(1,s) f(s, u_1(s)+Q_1 s^2+R_1,\dots,u_n(s)+Q_n s^2+R_n)ds \\
&\geq \lambda \int_\alpha^\beta G_n(1,s) f(s, r\|u_1\|_\infty,\dots,r\|u_n\|_\infty)ds
\end{aligned}
$$

Lemma 1: Proof

Pairing the above with the nondecreasing properties of f, it follows that

$$\begin{aligned}
\|T(u_1,\ldots,u_n)\| &\geq \|A_n(u_1,\ldots,u_n)\|_\infty \\
&\geq \lambda \int_0^1 G_n(1,s) f(s, u_1(s)+Q_1 s^2+R_1, \ldots, u_n(s)+Q_n s^2+R_n)ds \\
&\geq \lambda \int_\alpha^\beta G_n(1,s) f(s, r\|u_1\|_\infty, \ldots, r\|u_n\|_\infty)ds \\
&= \lambda r\|(u_1,\ldots,u_n)\| \int_\alpha^\beta G_n(1,s)\frac{f(s, r\|u_1\|_\infty, \ldots, r\|u_n\|_\infty)}{r\|(u_1,\ldots,u_n)\|}ds
\end{aligned}$$

Lemma 1: Proof

Pairing the above with the nondecreasing properties of f, it follows that

$$\begin{aligned}
\|T(u_1,\dots,u_n)\| &\geq \|A_n(u_1,\dots,u_n)\|_\infty \\
&\geq \lambda \int_0^1 G_n(1,s) f(s, u_1(s)+Q_1 s^2 + R_1, \dots, u_n(s) + Q_n s^2 + R_n) ds \\
&\geq \lambda \int_\alpha^\beta G_n(1,s) f(s, r\|u_1\|_\infty, \dots, r\|u_n\|_\infty) ds \\
&= \lambda r \|(u_1,\dots,u_n)\| \int_\alpha^\beta G_n(1,s) \frac{f(s, r\|u_1\|_\infty, \dots, r\|u_n\|_\infty)}{r\|(u_1,\dots,u_n)\|} ds \\
&\geq \lambda K r \|(u_1,\dots,u_n)\| \int_\alpha^\beta G_n(1,s) ds
\end{aligned}$$

Lemma 1: Proof

Pairing the above with the nondecreasing properties of f, it follows that

$$\begin{aligned}
\|T(u_1,\ldots,u_n)\| &\geq \|A_n(u_1,\ldots,u_n)\|_\infty \\
&\geq \lambda \int_0^1 G_n(1,s) f(s, u_1(s)+Q_1 s^2 + R_1, \ldots, u_n(s)+Q_n s^2 + R_n) ds \\
&\geq \lambda \int_\alpha^\beta G_n(1,s) f(s, r\|u_1\|_\infty, \ldots, r\|u_n\|_\infty) ds \\
&= \lambda r \|(u_1,\ldots,u_n)\| \int_\alpha^\beta G_n(1,s) \frac{f(s, r\|u_1\|_\infty, \ldots, r\|u_n\|_\infty)}{r\|(u_1,\ldots,u_n)\|} ds \\
&\geq \lambda K r \|(u_1,\ldots,u_n)\| \int_\alpha^\beta G_n(1,s) ds \\
&\geq \Lambda K r \|(u_1,\ldots,u_n)\| \int_\alpha^\beta G_n(1,s) ds
\end{aligned}$$

Lemma 1: Proof

Pairing the above with the nondecreasing properties of f, it follows that

$$\begin{aligned}
\|T(u_1,\dots,u_n)\| &\geq \|A_n(u_1,\dots,u_n)\|_\infty \\
&\geq \lambda \int_0^1 G_n(1,s) f(s, u_1(s)+Q_1 s^2+R_1,\dots,u_n(s)+Q_n s^2+R_n)ds \\
&\geq \lambda \int_\alpha^\beta G_n(1,s) f(s, r\|u_1\|_\infty,\dots,r\|u_n\|_\infty)ds \\
&= \lambda r\|(u_1,\dots,u_n)\| \int_\alpha^\beta G_n(1,s) \frac{f(s, r\|u_1\|_\infty,\dots,r\|u_n\|_\infty)}{r\|(u_1,\dots,u_n)\|} ds \\
&\geq \lambda K r\|(u_1,\dots,u_n)\| \int_\alpha^\beta G_n(1,s)ds \\
&\geq \Lambda K r\|(u_1,\dots,u_n)\| \int_\alpha^\beta G_n(1,s)ds \\
&\geq \|(u_1,\dots,u_n)\|
\end{aligned}$$

Lemma 1: Proof

Pairing the above with the nondecreasing properties of f, it follows that

$$\begin{aligned}
\|T(u_1,\ldots,u_n)\| &\geq \|A_n(u_1,\ldots,u_n)\|_\infty \\
&\geq \lambda \int_0^1 G_n(1,s) f(s, u_1(s) + Q_1 s^2 + R_1, \ldots, u_n(s) + Q_n s^2 + R_n) ds \\
&\geq \lambda \int_\alpha^\beta G_n(1,s) f(s, r\|u_1\|_\infty, \ldots, r\|u_n\|_\infty) ds \\
&= \lambda r \|(u_1,\ldots,u_n)\| \int_\alpha^\beta G_n(1,s) \frac{f(s, r\|u_1\|_\infty, \ldots, r\|u_n\|_\infty)}{r\|(u_1,\ldots,u_n)\|} ds \\
&\geq \lambda K r \|(u_1,\ldots,u_n)\| \int_\alpha^\beta G_n(1,s) ds \\
&\geq \Lambda K r \|(u_1,\ldots,u_n)\| \int_\alpha^\beta G_n(1,s) ds \\
&\geq \|(u_1,\ldots,u_n)\|
\end{aligned}$$

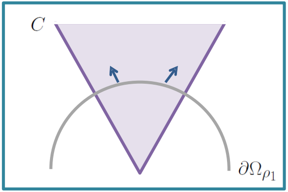
for $\lambda \geq \Lambda$, which completes the proof.

Lemma 2

Lemma 2. *Fix* $\Lambda > 0$, *and suppose* (H0) *and* (H1) *hold. Then, for every* $\lambda \geq \Lambda$ *and* $(Q_1, \ldots, Q_n, R_1, \ldots, R_n) \in [0, \infty)^{2n}$, *there exists positive* $\rho_1 = \rho_1(\Lambda, Q_1, \ldots, Q_n, R_1, \ldots, R_n)$ *such that, for every* $\rho \in (0, \rho_1]$, *we have*

$$\|T(u_1, \ldots, u_n)\| \geq \|(u_1, \ldots, u_n)\|$$

for each $(u_1, \ldots, u_n) \in C \cap \partial\Omega_\rho$.



Lemma 3: Setup

So far, we have found subsets $C \cap \partial\Omega_{\rho^*}$ and $C \cap \partial\Omega_{\rho_1}$ on which

$$\|T(u_1, \ldots, u_n)\| \geq \|(u_1, \ldots, u_n)\|.$$

Lemma 3: Setup

So far, we have found subsets $C \cap \partial\Omega_{\rho^*}$ and $C \cap \partial\Omega_{\rho_1}$ on which

$$\|T(u_1, \ldots, u_n)\| \geq \|(u_1, \ldots, u_n)\|.$$

Now, suppose we were to find $\rho_2 \in (0, \rho^*)$ such that

$$\|T(u_1, \ldots, u_n)\| \leq \|(u_1, \ldots, u_n)\|$$

for all $(u_1, \ldots, u_n) \in C \cap \partial\Omega_{\rho_2}$.

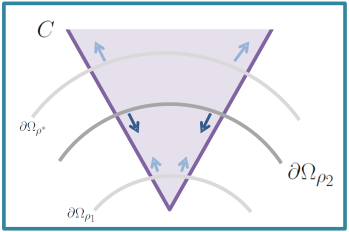
Lemma 3: Setup

Then, because Lemma 2 holds for **ALL** positive $\rho \leq \rho_1$, the Guo-Krosnoselskii Fixed Point Theorem would be satisfied twice:

Lemma 3: Setup

Then, because Lemma 2 holds for **ALL** positive $\rho \leq \rho_1$, the Guo-Krosnoselskii Fixed Point Theorem would be satisfied twice:

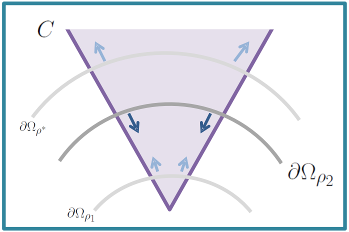
- Once over $C \cap (\overline{\Omega}_{\rho^*} - \Omega_{\rho_2})$ via the expansion form of the theorem.



Lemma 3: Setup

Then, because Lemma 2 holds for **ALL** positive $\rho \leq \rho_1$, the Guo-Krosnoselskii Fixed Point Theorem would be satisfied twice:

- Once over $C \cap (\overline{\Omega}_{\rho^*} - \Omega_{\rho_2})$ via the expansion form of the theorem.
- A second time over $C \cap (\overline{\Omega}_{\rho_2} - \Omega_{\rho_1})$ by the compression form of the theorem.



Lemma 3: Hypotheses

We find exactly such a ρ_2 in Lemma 3. The following hypotheses will be needed:

Lemma 3: Hypotheses

We find exactly such a ρ_2 in Lemma 3. The following hypotheses will be needed:

(H0) For $i = 1, 2, \ldots, n-1$, the functions $f, g_i : [0,1] \times [0,\infty)^n \to [0,\infty)$ are continuous and nondecreasing in their last n variables.

Lemma 3: Hypotheses

We find exactly such a ρ_2 in Lemma 3. The following hypotheses will be needed:

(H0) For $i = 1, 2, \ldots, n-1$, the functions $f, g_i : [0,1] \times [0,\infty)^n \to [0,\infty)$ are continuous and nondecreasing in their last n variables.

(H2) Let $z = \sum_{i=1}^{n} x_i$. Then

$$\lim_{z \to 0^+} \frac{f(t, x_1, \ldots, x_n)}{z} = 0$$

uniformly for $t \in [0,1]$.

Lemma 3: Hypotheses

We find exactly such a ρ_2 in Lemma 3. The following hypotheses will be needed:

(H0) For $i = 1, 2, \ldots, n-1$, the functions $f, g_i : [0,1] \times [0,\infty)^n \to [0,\infty)$ are continuous and nondecreasing in their last n variables.

(H2) Let $z = \sum_{i=1}^{n} x_i$. Then

$$\lim_{z \to 0^+} \frac{f(t, x_1, \ldots, x_n)}{z} = 0$$

uniformly for $t \in [0,1]$.

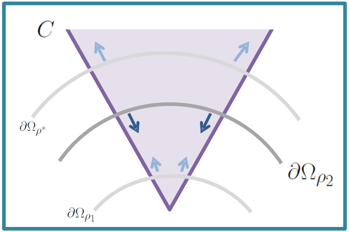
(H3) There exists $0 < \zeta_i < \frac{2M_iN_i}{\alpha_i(\delta_i+\beta_i)}$ and $q_i > 0$ such that, for all $(\overline{x}_1, \ldots, \overline{x}_n) \in [0,\infty)^n$ with $0 < \sum_{j=1}^{n} \overline{x}_j < q_i$, we have $g_i(t, \overline{x}_1, \ldots, \overline{x}_n) \le \zeta_i \sum_{j=1}^{n} \overline{x}_j$ for each $t \in [0,1]$ and $i = 1, 2, \ldots, n-1$.

Lemma 3

Lemma 3. *Suppose* (H0), (H2), *and* (H3) *hold, and let* $\rho^* > 0$ *be fixed. Then given* $\lambda > 0$*, there exists* $\rho_2 \in (0, \rho^*)$ *and* $\overline{\zeta} > 0$ *such that for every* $(Q_1, \ldots, Q_n, R_1, \ldots, R_n) \in [0, \infty)^{2n}$ *with* $\sum_{i=1}^n (Q_i + R_i) < \overline{\zeta}$*, we have*

$$\|T(u_1, \ldots, u_n)\| \leq \|(u_1, \ldots, u_n)\|$$

for each $(u_1, \ldots, u_n) \in C \cap \partial\Omega_{\rho_2}$.



Lemma 3: Proof

Given $\lambda > 0$, pick $\epsilon > 0$ so that $\lambda\epsilon < \frac{M_n N_n}{\alpha_n(\delta_n+\beta_n)}$.

Lemma 3: Proof

Given $\lambda > 0$, pick $\epsilon > 0$ so that $\lambda\epsilon < \frac{M_n N_n}{\alpha_n(\delta_n+\beta_n)}$.

From (H2), there exists $\overline{\rho}_2 \in (0, \rho^*)$ such that for $\sum_{i=1}^n x_i = \overline{\rho}_2$ with $(x_1, \dots, x_n) \in [0, \infty)^n$ and for $\sum_{i=1}^n (Q_i + R_i) \leq \overline{\rho}_2$, we have

$$f(t, x_1 + Q_1 + R_1, \dots, x_n + Q_n + R_n) \leq \epsilon\left[(x_1 + Q_1 + R_1) + \cdots + (x_n + Q_n + R_n)\right]$$

for $t \in [0, 1]$.

Lemma 3: Proof

Also, by (H3), there exists $\zeta_i > 0$ satisfying $\zeta_i < \frac{2M_iN_i}{\alpha_i(\delta_i+\beta_i)}$ and there exists $q_i > 0$ such that, for

$$(x_1 + Q_1 + R_1, \ldots, x_n + Q_n + R_n) \in [0, \infty)^n$$

with $\sum_{j=1}^{n} (x_j + Q_j + R_j) < q_i$, we have

$$g_i(t, x_1 + Q_1 + R_1, \ldots, x_n + Q_n + R_n) \leq \zeta_i \left[(x_1 + Q_1 + R_1) + \cdots + (x_n + Q_n + R_n)\right]$$

for $t \in [0, 1]$ and $i = 1, 2, \ldots, n-1$.

Lemma 3: Proof

Set $q = \min\{q_1, \ldots, q_{n-1}\}$, and let $0 < \rho_2 < \min\{q/2, \overline{\rho}_2\}$. Take $(u_1, \ldots, u_n) \in C \cap \partial\Omega_{\rho_2}$, and $\sum_{i=1}^{n} (Q_i + R_i) \leq \rho_2$. Then, by (H0) and above, we have

Lemma 3: Proof

Set $q = \min\{q_1, \ldots, q_{n-1}\}$, and let $0 < \rho_2 < \min\{q/2, \overline{\rho}_2\}$. Take $(u_1, \ldots, u_n) \in C \cap \partial\Omega_{\rho_2}$, and $\sum_{i=1}^{n}(Q_i + R_i) \leq \rho_2$. Then, by (H0) and above, we have

$$A_n(u_1, \ldots, u_n) = \lambda \int_0^1 G_n(t,s) f(s, u_1(s) + Q_1 s^2 + R_1, \ldots, u_n(s) + Q_n s^2 + R_n) ds$$

Lemma 3: Proof

Set $q = \min\{q_1, \ldots, q_{n-1}\}$, and let $0 < \rho_2 < \min\{q/2, \overline{\rho}_2\}$. Take $(u_1, \ldots, u_n) \in C \cap \partial\Omega_{\rho_2}$, and $\sum_{i=1}^{n} (Q_i + R_i) \leq \rho_2$. Then, by (H0) and above, we have

$$\begin{aligned} A_n(u_1, \ldots, u_n) &= \lambda \int_0^1 G_n(t,s) f(s, u_1(s) + Q_1 s^2 + R_1, \ldots, u_n(s) + Q_n s^2 + R_n) ds \\ &\leq \lambda \int_0^1 G_n(t,s) f(s, \|u_1\|_\infty + Q_1 + R_1, \ldots, \|u_n\|_\infty + Q_n + R_n) ds \end{aligned}$$

Lemma 3: Proof

Set $q = \min\{q_1, \ldots, q_{n-1}\}$, and let $0 < \rho_2 < \min\{q/2, \overline{\rho}_2\}$. Take $(u_1, \ldots, u_n) \in C \cap \partial\Omega_{\rho_2}$, and $\sum_{i=1}^{n}(Q_i + R_i) \leq \rho_2$. Then, by (H0) and above, we have

$$\begin{aligned} A_n(u_1, \ldots, u_n) &= \lambda \int_0^1 G_n(t,s) f(s, u_1(s) + Q_1 s^2 + R_1, \ldots, u_n(s) + Q_n s^2 + R_n) ds \\ &\leq \lambda \int_0^1 G_n(t,s) f(s, \|u_1\|_\infty + Q_1 + R_1, \ldots, \|u_n\|_\infty + Q_n + R_n) ds \\ &\leq \lambda\epsilon \left[\|(u_1, \ldots, u_n)\| + \sum_{i=1}^{n}(Q_i + R_i) \right] \int_0^1 G_n(t,s) ds \end{aligned}$$

Lemma 3: Proof

Set $q = \min\{q_1, \ldots, q_{n-1}\}$, and let $0 < \rho_2 < \min\{q/2, \overline{\rho}_2\}$. Take $(u_1, \ldots, u_n) \in C \cap \partial\Omega_{\rho_2}$, and $\sum_{i=1}^{n}(Q_i + R_i) \leq \rho_2$. Then, by (H0) and above, we have

$$
\begin{aligned}
A_n(u_1, \ldots, u_n) &= \lambda \int_0^1 G_n(t,s) f(s, u_1(s) + Q_1 s^2 + R_1, \ldots, u_n(s) + Q_n s^2 + R_n) ds \\
&\leq \lambda \int_0^1 G_n(t,s) f(s, \|u_1\|_\infty + Q_1 + R_1, \ldots, \|u_n\|_\infty + Q_n + R_n) ds \\
&\leq \lambda \epsilon \left[\|(u_1, \ldots, u_n)\| + \sum_{i=1}^{n} (Q_i + R_i) \right] \int_0^1 G_n(t,s) ds \\
&\leq 2\lambda\epsilon \|(u_1, \ldots, u_n)\| \int_0^1 G_n(t,s) ds
\end{aligned}
$$

Lemma 3: Proof

Set $q = \min\{q_1, \ldots, q_{n-1}\}$, and let $0 < \rho_2 < \min\{q/2, \overline{\rho}_2\}$. Take $(u_1, \ldots, u_n) \in C \cap \partial\Omega_{\rho_2}$, and $\sum_{i=1}^{n}(Q_i + R_i) \leq \rho_2$. Then, by (H0) and above, we have

$$\begin{aligned}
A_n(u_1, \ldots, u_n) &= \lambda \int_0^1 G_n(t,s) f(s, u_1(s) + Q_1 s^2 + R_1, \ldots, u_n(s) + Q_n s^2 + R_n) ds \\
&\leq \lambda \int_0^1 G_n(t,s) f(s, \|u_1\|_\infty + Q_1 + R_1, \ldots, \|u_n\|_\infty + Q_n + R_n) ds \\
&\leq \lambda\epsilon \left[\|(u_1, \ldots, u_n)\| + \sum_{i=1}^{n}(Q_i + R_i) \right] \int_0^1 G_n(t,s) ds \\
&\leq 2\lambda\epsilon \|(u_1, \ldots, u_n)\| \int_0^1 G_n(t,s) ds \\
&\leq \lambda\epsilon \frac{\alpha_n(\delta_n + \beta_n)}{M_n N_n} \|(u_1, \ldots, u_n)\|
\end{aligned}$$

Lemma 3: Proof

Set $q = \min\{q_1, \ldots, q_{n-1}\}$, and let $0 < \rho_2 < \min\{q/2, \overline{\rho}_2\}$. Take $(u_1, \ldots, u_n) \in C \cap \partial\Omega_{\rho_2}$, and $\sum_{i=1}^{n}(Q_i + R_i) \leq \rho_2$. Then, by (H0) and above, we have

$$
\begin{aligned}
A_n(u_1, \ldots, u_n) &= \lambda \int_0^1 G_n(t,s) f(s, u_1(s) + Q_1 s^2 + R_1, \ldots, u_n(s) + Q_n s^2 + R_n) ds \\
&\leq \lambda \int_0^1 G_n(t,s) f(s, \|u_1\|_\infty + Q_1 + R_1, \ldots, \|u_n\|_\infty + Q_n + R_n) ds \\
&\leq \lambda \epsilon \left[\|(u_1, \ldots, u_n)\| + \sum_{i=1}^{n} (Q_i + R_i) \right] \int_0^1 G_n(t,s) ds \\
&\leq 2\lambda\epsilon \|(u_1, \ldots, u_n)\| \int_0^1 G_n(t,s) ds \\
&\leq \lambda\epsilon \frac{\alpha_n(\delta_n + \beta_n)}{M_n N_n} \|(u_1, \ldots, u_n)\|
\end{aligned}
$$

for $t \in [0,1]$.

Lemma 3: Proof

To establish similar bounds for $A_1, \ldots, A_{n-1}$, note that

$$\sum_{i=1}^{n} \left(\|u_i\|_\infty + Q_i + R_i \right) \leq 2\rho_2 < q = \min\left\{q_1, \ldots, q_{n-1}\right\}.$$

Lemma 3: Proof

To establish similar bounds for $A_1, \ldots, A_{n-1}$, note that

$$\sum_{i=1}^{n} (\|u_i\|_\infty + Q_i + R_i) \leq 2\rho_2 < q = \min\{q_1, \ldots, q_{n-1}\}.$$

So,

$$g_i(t, \|u_1\|_\infty + Q_1 + R_1, \ldots \|u_n\|_\infty + Q_n + R_n) \leq \zeta_i \sum_{j=1}^{n} (\|u_j\|_\infty + Q_j + R_j)$$

for $i = 1, 2, \ldots, n-1$.

Lemma 3: Proof

Now, pick $\zeta' < 1$, and suppose $\sum_{i=1}^{n} (Q_i + R_i) < \zeta' \rho_2$. Set $\overline{\zeta} = \zeta' \rho_2$.

Lemma 3: Proof

Now, pick $\zeta' < 1$, and suppose $\sum_{i=1}^{n}(Q_i + R_i) < \zeta' \rho_2$. Set $\overline{\zeta} = \zeta' \rho_2$.

Then it follows by (H0) and above that

Lemma 3: Proof

Now, pick $\zeta' < 1$, and suppose $\sum_{i=1}^{n}(Q_i + R_i) < \zeta'\rho_2$. Set $\overline{\zeta} = \zeta'\rho_2$.

Then it follows by (H0) and above that

$$A_i(u_1, \ldots, u_3) = \int_0^1 G_i(t,s)g_i(s, u_1(s) + Q_1 s^2 + R_1, \ldots, u_n(s) + Q_n s^2 + R_n)ds$$

Lemma 3: Proof

Now, pick $\zeta' < 1$, and suppose $\sum_{i=1}^{n}(Q_i + R_i) < \zeta'\rho_2$. Set $\overline{\zeta} = \zeta'\rho_2$.

Then it follows by (H0) and above that

$$\begin{aligned} A_i(u_1, \ldots, u_3) &= \int_0^1 G_i(t,s) g_i(s, u_1(s) + Q_1 s^2 + R_1, \ldots, u_n(s) + Q_n s^2 + R_n) ds \\ &\leq \zeta_i \left[\|(u_1, \ldots, u_n)\| + \sum_{j=1}^{n} (Q_j + R_j) \right] \int_0^1 G_i(t,s) ds \end{aligned}$$

Lemma 3: Proof

Now, pick $\zeta' < 1$, and suppose $\sum_{i=1}^{n}(Q_i + R_i) < \zeta' \rho_2$. Set $\overline{\zeta} = \zeta' \rho_2$.

Then it follows by (H0) and above that

$$\begin{aligned} A_i(u_1, \ldots, u_3) &= \int_0^1 G_i(t,s) g_i(s, u_1(s) + Q_1 s^2 + R_1, \ldots, u_n(s) + Q_n s^2 + R_n) ds \\ &\leq \zeta_i \left[\|(u_1, \ldots, u_n)\| + \sum_{j=1}^{n} (Q_j + R_j) \right] \int_0^1 G_i(t,s) ds \\ &\leq \zeta_i \left(1 + \zeta'\right) \|(u_1, \ldots, u_n)\| \int_0^1 G_i(t,s) ds \end{aligned}$$

Lemma 3: Proof

Now, pick $\zeta' < 1$, and suppose $\sum_{i=1}^{n}(Q_i + R_i) < \zeta'\rho_2$. Set $\overline{\zeta} = \zeta'\rho_2$.

Then it follows by (H0) and above that

$$
\begin{aligned}
A_i(u_1, \ldots, u_3) &= \int_0^1 G_i(t,s) g_i(s, u_1(s) + Q_1 s^2 + R_1, \ldots, u_n(s) + Q_n s^2 + R_n) ds \\
&\leq \zeta_i \left[\|(u_1, \ldots, u_n)\| + \sum_{j=1}^{n} (Q_j + R_j) \right] \int_0^1 G_i(t,s) ds \\
&\leq \zeta_i (1 + \zeta') \|(u_1, \ldots, u_n)\| \int_0^1 G_i(t,s) ds \\
&\leq \zeta_i (1 + \zeta') \frac{\alpha_i (\delta_i + \beta_i)}{2 M_i N_i} \|(u_1, \ldots, u_n)\|
\end{aligned}
$$

Lemma 3: Proof

Now, pick $\zeta' < 1$, and suppose $\sum_{i=1}^{n}(Q_i + R_i) < \zeta'\rho_2$. Set $\overline{\zeta} = \zeta'\rho_2$.

Then it follows by (H0) and above that

$$\begin{aligned} A_i(u_1, \ldots, u_3) &= \int_0^1 G_i(t,s)g_i(s, u_1(s) + Q_1 s^2 + R_1, \ldots, u_n(s) + Q_n s^2 + R_n)ds \\ &\leq \zeta_i \left[\|(u_1, \ldots, u_n)\| + \sum_{j=1}^{n}(Q_j + R_j)\right] \int_0^1 G_i(t,s)ds \\ &\leq \zeta_i (1 + \zeta') \|(u_1, \ldots, u_n)\| \int_0^1 G_i(t,s)ds \\ &\leq \zeta_i (1 + \zeta') \frac{\alpha_i(\delta_i + \beta_i)}{2M_i N_i} \|(u_1, \ldots, u_n)\| \end{aligned}$$

for $t \in [0,1]$ and $i = 1, 2, \ldots, n-1$.

Lemma 3: Proof

Thus,

$$\|T(u_1,\dots,u_n)\| \le \left[\lambda\epsilon\frac{\alpha_n(\delta_n+\beta_n)}{M_nN_n} + (1+\zeta')\sum_{i=1}^{n-1}\zeta_i\frac{\alpha_i(\delta_i+\beta_i)}{2M_iN_i}\right]\|(u_1,\dots,u_n)\|$$

for $(u_1,\dots,u_n)\in C\cap\Omega_{\rho_2}$ and $(Q_1,\dots,Q_n,R_1,\dots,R_n)\in[0,\infty)^{2n}$ with

$$\sum_{i=1}^{n}(Q_i+R_i)<\overline{\zeta}.$$

Lemma 3: Proof

Thus,

$$\|T(u_1,\ldots,u_n)\| \leq \left[\lambda\epsilon\frac{\alpha_n(\delta_n+\beta_n)}{M_nN_n} + (1+\zeta')\sum_{i=1}^{n-1}\zeta_i\frac{\alpha_i(\delta_i+\beta_i)}{2M_iN_i}\right]\|(u_1,\ldots,u_n)\|$$

for $(u_1,\ldots,u_n)\in C\cap\Omega_{\rho_2}$ and $(Q_1,\ldots,Q_n,R_1,\ldots,R_n)\in[0,\infty)^{2n}$ with

$$\sum_{i=1}^{n}(Q_i+R_i)<\overline{\zeta}.$$

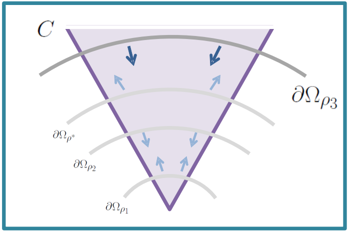
Picking ϵ and ζ' small enough gives the desired result.

Lemma 4: Setup

A third (and final) fixed point can be obtained by establishing a contraction estimate for T on $C \cap \partial\Omega_{\rho_3}$, where $\rho_3 > \rho^*$, and by utilizing the estimate of Lemma 1.

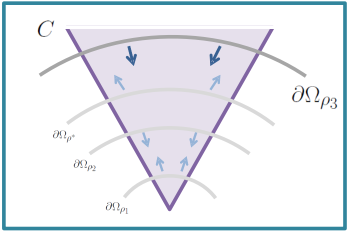
Lemma 4: Setup

A third (and final) fixed point can be obtained by establishing a contraction estimate for T on $C \cap \partial\Omega_{\rho_3}$, where $\rho_3 > \rho^*$, and by utilizing the estimate of Lemma 1.



Lemma 4: Setup

A third (and final) fixed point can be obtained by establishing a contraction estimate for T on $C \cap \partial\Omega_{\rho_3}$, where $\rho_3 > \rho^*$, and by utilizing the estimate of Lemma 1.



Notice that, in this case, we would apply the compression form of the Guo-Krasnosel'skii Fixed Point Theorem on the set $C \cap (\overline{\Omega}_{\rho_3} - \Omega_{\rho^*})$.

Lemma 4: Hypotheses

This is the purpose of Lemma 4. The following hypotheses will be required:

Lemma 4: Hypotheses

This is the purpose of Lemma 4. The following hypotheses will be required:

(H0) For $i = 1, 2, \ldots, n-1$, the functions $f, g_i : [0,1] \times [0,\infty)^n \to [0,\infty)$ are continuous and nondecreasing in their last n variables.

Lemma 4: Hypotheses

This is the purpose of Lemma 4. The following hypotheses will be required:

(H0) For $i = 1, 2, \ldots, n-1$, the functions $f, g_i : [0,1] \times [0,\infty)^n \to [0,\infty)$ are continuous and nondecreasing in their last n variables.

(H4) Let $z = \sum_{i=1}^{n} x_i$. Then

$$\lim_{z\to\infty} \frac{f(t, x_1, \ldots, x_n)}{z} = 0$$

uniformly for $t \in [0,1]$.

Lemma 4: Hypotheses

This is the purpose of Lemma 4. The following hypotheses will be required:

(H0) For $i = 1, 2, \ldots, n-1$, the functions $f, g_i : [0,1] \times [0,\infty)^n \to [0,\infty)$ are continuous and nondecreasing in their last n variables.

(H4) Let $z = \sum_{i=1}^{n} x_i$. Then

$$\lim_{z\to\infty} \frac{f(t, x_1, \ldots, x_n)}{z} = 0$$

uniformly for $t \in [0,1]$.

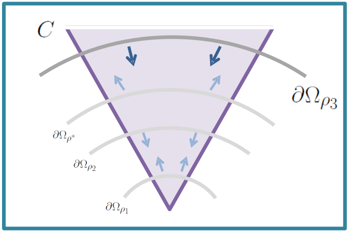
(H5) There exists $0 < \theta_i < \frac{2M_iN_i}{\alpha_i(\delta_i+\beta_i)}$ and $r_i > 0$ such that, for all $(\overline{x}_1, \ldots, \overline{x}_n) \in [0,\infty)^n$ with $\sum_{j=1}^{n} \overline{x}_j > r_i$, we have $g_i(t, \overline{x}_1, \ldots, \overline{x}_n) \le \theta_i \sum_{j=1}^{n} \overline{x}_j$ for each $t \in [0,1]$ and $i = 1, 2, \ldots, n-1$.

Lemma 4

Lemma 4. *Let* $(Q_1, \ldots, Q_n, R_1, \ldots, R_n) \in [0, \infty)^{2n}$, *and suppose* $\sum_{i=1}^{n} (Q_i + R_i) < \overline{\zeta}$, *where* $\overline{\zeta} > 0$ *is given. Suppose further that assumptions* (H0), (H4), *and* (H5) *hold. Then, for every* $\lambda > 0$, *there exists* $\rho_3 = \rho_3(\overline{\zeta}, \lambda)$ *such that for every* $\rho \geq \rho_3$, *we have*

$$\|T(u_1, \ldots, u_n)\| \leq \|(u_1, \ldots, u_n)\|$$

for each $(u_1, \ldots, u_n) \in C \cap \partial\Omega_\rho$.

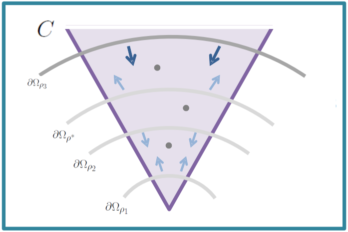


Main Results

4. Apply the Guo-Krasnosel'skii Fixed Point Theorem three times to show the existence of **at least** three fixed points of T and, hence, at least three positive solutions to the transformed problem. The existence of multiple positive solutions to the original problem can then be established as a corollary.

Theorem 1

Theorem 1. *Suppose hypotheses* (H0)–(H5) *are satisfied for functions* $f, g_1, g_2, \ldots, g_{n-1}$. *Suppose additionally that* $\alpha_i > \gamma_i \geq 2\delta_i > \delta_i > \beta_i > 0$ *for* $i = 1, 2, \ldots, n$. *Then there exists* $\Lambda > 0$ *such that given* $\lambda \geq \Lambda$, *there exists* $\overline{\zeta} > 0$ *such that for every* $a_1, a_2, \ldots, a_n > 0$ *satisfying* $\sum_{i=1}^{n} \frac{a_i}{2\delta_i}\left[1 - \frac{2\delta_i - \gamma_i}{\alpha_i - \gamma_i}\right] < \overline{\zeta}$ *and every* $(Q_1, \ldots, Q_n, R_1, \ldots, R_n) \in [0, \infty)^{2n}$ *satisfying* $\sum_{i=1}^{n} (Q_i + R_i) < \overline{\zeta}$, *the system* (7)–(9) *has at least three positive solutions.*



Corollary 1

By utilizing the one-one correspondence of (1)–(3) with (7)–(9) and the previous theorem, we can obtain an existence result for the original system.

Corollary 1

Corollary 1. *Suppose the following hypotheses are satisfied for a function* $h : [0,1] \times [0,\infty) \times (-\infty, 0] \to [0,\infty)$*:*

Corollary 1

Corollary 1. *Suppose the following hypotheses are satisfied for a function* $h : [0,1] \times [0,\infty) \times (-\infty, 0] \to [0,\infty)$*:*

(H0′) h is continuous, nondecreasing in its $(2j)$th variables, and nonincreasing in its $(2j+1)$th variables for $j = 1, 2, \ldots, n$.

Corollary 1

Corollary 1. *Suppose the following hypotheses are satisfied for a function* $h : [0,1] \times [0,\infty) \times (-\infty, 0] \to [0,\infty)$*:*

(H0′) h is continuous, nondecreasing in its $(2j)$th variables, and nonincreasing in its $(2j+1)$th variables for $j = 1, 2, \ldots, n$.

(H1′) There exists $\alpha, \beta \in (0,1)$, $\alpha < \beta$, such that, given $(x_1, \ldots, x_n) \in \prod_{i=1}^{n} (-1)^{i-1}[0,\infty)$ with $\sum_{i=1}^{n} x_i \neq 0$, there exists $\kappa > 0$ such that $h(t, x_1, x_2, \ldots, x_n) > \kappa$ for $t \in [\alpha, \beta]$.

Corollary 1

Corollary 1. *Suppose the following hypotheses are satisfied for a function* $h : [0,1] \times [0,\infty) \times (-\infty, 0] \to [0,\infty)$*:*

(H0′) h is continuous, nondecreasing in its $(2j)$th variables, and nonincreasing in its $(2j+1)$th variables for $j = 1, 2, \ldots, n$.

(H1′) There exists $\alpha, \beta \in (0,1)$, $\alpha < \beta$, such that, given $(x_1, \ldots, x_n) \in \prod_{i=1}^{n}(-1)^{i-1}[0,\infty)$ with $\sum_{i=1}^{n} x_i \neq 0$, there exists $\kappa > 0$ such that $h(t, x_1, x_2, \ldots, x_n) > \kappa$ for $t \in [\alpha, \beta]$.

(H2′) Let $z = \sum_{i=1}^{n}(-1)^{i-1}x_i > 0$. Then $\lim_{z \to 0^+} \frac{h(t, x_1, \ldots, x_n)}{z} = 0$ uniformly for $t \in [0,1]$.

Corollary 1

Corollary 1. *Suppose the following hypotheses are satisfied for a function* $h : [0,1] \times [0,\infty) \times (-\infty,0] \to [0,\infty)$*:*

(H0′) h is continuous, nondecreasing in its $(2j)$th variables, and nonincreasing in its $(2j+1)$th variables for $j = 1, 2, \ldots, n$.

(H1′) There exists $\alpha, \beta \in (0,1)$, $\alpha < \beta$, such that, given $(x_1, \ldots, x_n) \in \prod_{i=1}^{n}(-1)^{i-1}[0,\infty)$ with $\sum_{i=1}^{n} x_i \neq 0$, there exists $\kappa > 0$ such that $h(t, x_1, x_2, \ldots, x_n) > \kappa$ for $t \in [\alpha, \beta]$.

(H2′) Let $z = \sum_{i=1}^{n}(-1)^{i-1}x_i > 0$. Then $\lim_{z\to 0^+} \frac{h(t,x_1,\ldots,x_n)}{z} = 0$ uniformly for $t \in [0,1]$.

(H4′) Let $z = \sum_{i=1}^{n}(-1)^{i-1}x_i > 0$. Then $\lim_{z\to \infty} \frac{h(t,x_1,\ldots,x_n)}{z} = 0$ uniformly for $t \in [0,1]$.

Corollary 1

Suppose also that $\alpha_i > \gamma_i \geq 2\delta_i > \delta_i > \beta_i > 0$ *for* $i = 1, 2, \ldots, n$.

Corollary 1

Suppose also that $\alpha_i > \gamma_i \geq 2\delta_i > \delta_i > \beta_i > 0$ *for* $i = 1, 2, \ldots, n$.

Then there exists $\Lambda > 0$ *such that given* $\lambda \geq \Lambda$, *there exists* $\overline{\zeta} > 0$ *such that, for every* $a_1, a_2, \ldots, a_n > 0$ *that satisfies the properties that after setting* $Q_i = \frac{a}{2\delta_i}$ *and* $R_i = -\frac{a(2\delta_i - \gamma_i)}{2\delta_i(\alpha_i - \gamma_i)}$ *for* $i = 1, 2 \ldots, n$ *we obtain*

Corollary 1

Suppose also that $\alpha_i > \gamma_i \geq 2\delta_i > \delta_i > \beta_i > 0$ *for* $i = 1, 2, \ldots, n$.

Then there exists $\Lambda > 0$ *such that given* $\lambda \geq \Lambda$, *there exists* $\overline{\zeta} > 0$ *such that, for every* $a_1, a_2, \ldots, a_n > 0$ *that satisfies the properties that after setting* $Q_i = \frac{a}{2\delta_i}$ *and* $R_i = -\frac{a(2\delta_i - \gamma_i)}{2\delta_i(\alpha_i - \gamma_i)}$ *for* $i = 1, 2 \ldots, n$ *we obtain*

$$0 < \sum_{i=1}^{n} (Q_i + R_i) < \overline{\zeta}$$

and

Corollary 1

Suppose also that $\alpha_i > \gamma_i \geq 2\delta_i > \delta_i > \beta_i > 0$ *for* $i = 1, 2, \ldots, n$.

Then there exists $\Lambda > 0$ *such that given* $\lambda \geq \Lambda$, *there exists* $\bar{\zeta} > 0$ *such that, for every* $a_1, a_2, \ldots, a_n > 0$ *that satisfies the properties that after setting* $Q_i = \frac{a}{2\delta_i}$ *and* $R_i = -\frac{a(2\delta_i - \gamma_i)}{2\delta_i(\alpha_i - \gamma_i)}$ *for* $i = 1, 2 \ldots, n$ *we obtain*

$$0 < \sum_{i=1}^{n} (Q_i + R_i) < \bar{\zeta}$$

and

$$Q_n + R_n < \frac{2(\delta_i - \beta_i)(\alpha_i - \gamma_i)}{\alpha_i(\delta_i + \beta_i)} \sum_{j=1}^{n} (Q_j + R_j), \quad i = 1, 2, \ldots, n-1,$$

Corollary 1

Suppose also that $\alpha_i > \gamma_i \geq 2\delta_i > \delta_i > \beta_i > 0$ *for* $i = 1, 2, \ldots, n$.

Then there exists $\Lambda > 0$ *such that given* $\lambda \geq \Lambda$, *there exists* $\bar{\zeta} > 0$ *such that, for every* $a_1, a_2, \ldots, a_n > 0$ *that satisfies the properties that after setting* $Q_i = \frac{a}{2\delta_i}$ *and* $R_i = -\frac{a(2\delta_i - \gamma_i)}{2\delta_i(\alpha_i - \gamma_i)}$ *for* $i = 1, 2 \ldots, n$ *we obtain*

$$0 < \sum_{i=1}^{n} (Q_i + R_i) < \bar{\zeta}$$

and

$$Q_n + R_n < \frac{2(\delta_i - \beta_i)(\alpha_i - \gamma_i)}{\alpha_i(\delta_i + \beta_i)} \sum_{j=1}^{n} (Q_j + R_j), \quad i = 1, 2, \ldots, n-1,$$

then the system (1)–(3) *has at least three positive solutions.*